RECONSTRUCTION OF HEAT FLUXES THROUGH DIFFERENTIAL TEMPERATURE MEASUREMENT BY THE METHOD OF INVERSE DYNAMIC SYSTEMS

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UDC 536.2

Analytic and asymptotic solutions are obtained for the problem of reconstructing the nonsteady boundary heat fluxes in a flat plate with constant thermophysical parameters. The properties of the well-posed nature of this inverse problem are investigated.

The method of inversion of linear dynamic systems (DS) was developed in the theory of automatic control and encoding for the solution of problems connected with the reconstruction of the input signals and control of the output signals of lumped DS [1, 2]. The essence of the method consists in the representation of the inverse system in a space of states. Along with the solution of the inverse problem, such a representation allows one to answer a number of questions of a qualitative character: to indicate the minimum information about the initial conditions and outside actions sufficient for the reconstruction of input signals, to separate the inverse DS into well-posed and ill-posed parts in the Hadamard sense, and to determine whether the well-posed part of the DS is stable in the Lyapunov sense.

The concept of invertibility of lumped DS is used in the monograph [3] in connection with the solution of inverse problems analyzed within the framework of differential-difference models of thermal systems.* Some procedures of inversion and conditions of invertibility of distributed DS are proposed in [4-6].

In the present article the method of inversion of DS is used to solve the well-known problem [7] of reconstructing nonsteady boundary heat fluxes from the results of differential temperature measurements. The proposed approach yields an analytic solution and answers the above-stated questions of a qualitative character. We note that in the construction of an inverse system, a non-self-adjoint, boundary-value problem with a nonclassical boundary condition arises [8], which we reduce to the successive solution of two simple, self-adjoint, boundary-value problems.

Under the assumption of constancy of the thermophysical properties, we consider an infinite flat plate subject to thermal action from one side ($x = s > 0$) and thermally insulated on the other ($x = 0$). The inverse problem consists in the reconstruction of the heat flux at the surface $x = s$ by measuring the temperature difference $T(s, t)-T(0, t)$. The corresponding DS $\Omega$ describing the direct problem has the form

$$\Omega:\begin{cases} T_t = aT_{xx} + f(x, t), \ T(x, 0) = T_0(x), \\ T_x(0, t) = 0, \ -\lambda T_x(s, t) = q(t), \\ y(t) = T(s, t) - T(0, t). \end{cases}$$

From the system point of view [9] this inverse problem can be interpreted as the problem of reconstructing the input q of the DS $\Omega$ from the output y. The volume of information about the initial state $T_0(x)$ and the internal source of power $\lambda/a\ f(x, t)$ which is needed to reconstruct q is of definite interest here.

To represent the inverse system $\Omega^{-1}$ in the space of states we transpose the input and output of the direct DS $\Omega$. As a result, we obtain

*Instead of the term "observability at entry," proposed in [3], we use the term "invertibility of DS," popular in the literature on system theory.

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 47, No. 3, pp. 469-474, September, 1984. Original article submitted May 5, 1983.

 0022-0841/84/4703-1098$08.50 

$$\Omega^{-1}:\begin{cases}T_t = aT_{xx} + f,\ T(x,\ 0) = T_0(x), & (1)\\ T_x(0,\ t) = 0,\ T(s,\ t) - T(0,\ t) = y(t), & (2)\\ q(t) = -\lambda T_x(s,\ t). & (3)\end{cases}$$

Now to calculate q we must determine the Green's function for the boundary-value problem (1), (2). Since the generating operator A of the problem (1), (2)

$$Aw = aw_{xx},\ w_x(0) = 0,\ w(s) - w(0) = 0$$

is not self-adjoint and the spectrum A is double, the construction of the Green's function is associated with certain difficulties. As shown in recent research [8], however, for a certain choice of the system of eigenfunctions and adjoint functions the operator A of the Green's function can be constructed using the method of separation of variables. We offer another approach, connected with the idea of reduction to the space of states and allowing one to reduce the solution of a non-self-adjoint problem to a successive analysis of two self-adjoint systems. We also note that the spectrum of the operator A is located on the non-positive part of the real axis. Consequently, the DS $\Omega^{-1}$ is stable in the Lyapunov sense.

We expand the space H of states of the DS $\Omega$ into a direct sum of two subspaces $H^+$ and $H^-$,

$$T^+(x,\ t) = \frac{T(x,\ t) + T(s - x,\ t)}{2},$$

$$T^-(x,\ t) = \frac{T(x,\ t) - T(s - x,\ t)}{2}. \quad (4)$$

In accordance with (4), $T^+$ is the even part of the function T relative to the axis x = s/2 and $T^-$ is its odd part. Substituting (4) into Eqs. (1), (2), and (3) and using the relations $T_x^+ \in H^-$, $T_x^- \in H^+$, and $T \equiv 0 \Longleftrightarrow \{T^+ \equiv 0,\ T^- \equiv 0\}$, we note that the DS $\Omega$ can be represented in the form of a parallel union of two DS, $\Omega^+$ and $\Omega^-$ (see Fig. 1a), where

$$\Omega^+:\begin{cases}T_t^+ = aT_{xx}^+ + f^+,\\ T^+(x,\ 0) = T_0^+(x),\\ -2\lambda T_x^+(s,\ t) = q(t),\\ T^+ \in H^+,\end{cases} \qquad \Omega^-:\begin{cases}T_t^- = aT_{xx}^- + f^-,\\ T^-(x,\ 0) = T_0^-(x),\\ -2\lambda T_x^-(s,\ t) = q(t),\\ y(t) = 2T^-(s,\ t),\\ T^- \in H^-.\end{cases}$$

The form of the block diagram of the inverse DS $\Omega^{-1}$ follows directly from Fig. 1a (see Fig. 1b). The generating operator of the inverse DS

$$(\Omega^-)^{-1}:\begin{cases}T_t^- = aT_{xx}^- + f^-,\ T^-(x,\ 0) = T_0^-(x), & (5)\\ 2T^-(s,\ t) = y(t),\ T^- \in H^-, & \\ q(t) = -2\lambda T_x^-(s,\ t) & (6)\end{cases}$$

already possesses the self-adjoint property in the space $H^-$, so that the Green's function of the boundary-value problem (5), (6) is easily calculated using the method of separation of variables. Moreover, the DS $(\Omega^-)^{-1}$ is a narrowing of the ordinary DS

$$\Psi:\begin{cases}T_t = aT_{xx} + f^-,\ T(x,\ 0) = T_0^-(x), & (7)\\ 2T(0,\ t) = -y(t),\ 2T(s,\ t) = y(t), & (8)\\ q(t) = -2\lambda T_x(s,\ t)\end{cases}$$

with boundary conditions of the first kind, to the space of states $H^-$. In other words, the equality

$$(\Omega^-)^{-1} = \Psi^- \quad (9)$$

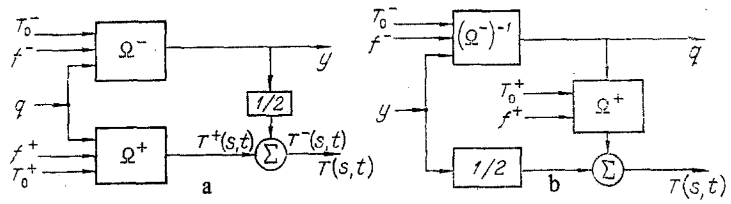


Fig. 1. Block diagrams of the dynamic systems Ω (a) and $\Omega^{-1}$ (b).

is satisfied. The Green's function of the boundary-value problem (7), (8) is known, so that the form of the Green's function* of the DS $\Psi^-$ follows quickly from Eq. (9):

$$\Psi^-(x,\ \xi,\ t)=\frac{2}{s}\sum_{k=1}^{\infty}\sin\frac{2k\pi x}{s}\sin\frac{2k\pi\xi}{s}\exp\left(-\left(\frac{2k\pi}{s}\right)^2 at\right).$$

Thus, the value of the heat-flux density in the plane x = s can be calculated from the equation

$$q(t)=-\frac{\partial}{\partial x}2\lambda\left(\int_0^s\Psi^-(x,\ \xi,\ t)T_0^-(\xi)\,d\xi+\int_0^t\int_0^s\Psi^-(x,\ \xi,\ t-\tau)f^-(\xi,\ \tau)\,d\xi d\tau+a\int_0^t\Psi_\xi^-(x,\ s,\ t-\tau)\,y(\tau)\,d\tau\right)\Big|_{x=s}. \quad (10)$$

Following the usual concepts of the well-posed nature of linear equations, we call the inverse DS $\Psi^-$ well-posed relative to the pair of Banach spaces M and N if the affiliation of the output q(·) to the space N follows from the condition of affiliation of an arbitrary input y(·) to the space M. The equivalence of the well-posed properties of the DS $\Psi^-$ relative to the pair of spaces M and N and the continuity of the input—output mapping acting from M into N follows from the well-known closed-graph theorem.

We demonstrate the well-posed nature of the DS $\Psi^-$ under the assumption that the input actions y(·) satisfy the Hölder condition $(M=h^\alpha[0,\ b])$: $|y(\tau')-y(\tau'')|<k|\tau'-\tau''|^\alpha$, $\frac{1}{2}<\alpha\leqslant 1$, $k>0$, $0\leqslant\tau'\leqslant\tau''\leqslant b$, $0<b<\infty$, while the output functions q(·) are estimated in accordance with the norm

$$\int_0^b|q(\tau)|\,d\tau$$

of the space of functions summable over the segment [0, b] (N = L[0, b]). For this we use the transformation

$$T^-=V_x^+,\ V^+\in H^+,\ V^+=\int_{\frac{s}{2}}^{x}T^-(\xi,\ t)\,d\xi+v(t)$$

of the space of states $H^-$ of the DS $\Psi^-$.

We set $v(t)=V_t^+\left(\frac{s}{2},\ t\right)-aV_{xx}^+\left(\frac{s}{2},\ t\right)$, and then we obtain the DS

$$\tilde{\Omega}^+:\begin{cases}V_t^+=aV_{xx}^++\tilde{f},\ V^+(x,\ 0)=V_0^+(x),\\ 2V_x^+(s,\ t)=y(t),\\ q(t)=-2\lambda V_{xx}^+(s,\ t)\equiv-\dfrac{2\lambda}{a}(V_t^+(s,\ t)-\tilde{f}(s,\ t)),\end{cases}$$

$$\tilde{f}=\int_{\frac{s}{2}}^{x}f^-(\xi,\ t)\,d\xi,\ V_0^+=\int_{\frac{s}{2}}^{x}T_0^-(\xi)\,d\xi,$$

*We shall designate the Green's function by the symbol for the corresponding dynamic system.

equivalent to the system $\Psi^-$ in the sense of input—output mapping. The Green's function of the DS $\tilde{\Omega}^+$ coincides with the Green's function of the DS $\Omega^+$ and has the form

$$\Omega^+(x,\ \xi,\ t) = \frac{1}{s}\left(1 + 2\sum_{k=1}^{\infty} \cos\frac{2k\pi x}{s} \cos\frac{2k\pi\xi}{s} \exp\left(-\left(\frac{2k\pi}{s}\right)^2 at\right)\right).$$

From this we get one more expression for the heat flux q:

$$q(t) = -\frac{2\lambda}{a}\frac{d}{dt}\left(\int_0^s \Omega^+(s,\ \xi,\ t)\,V_0^+(\xi)\,d\xi + \right.$$

$$\left. + \int_0^t\int_0^s \Omega^+(s,\ \xi,\ t-\tau)\,\tilde{f}(\xi,\ \tau)\,d\xi d\tau + a\int_0^t \Omega^+(s,\ s,\ t-\tau)\,y(\tau)\,d\tau\right) + \frac{2\lambda}{a}\tilde{f}(s,\ t). \tag{11}$$

The function

$$\Omega^+(s,\ s,\ t) = \frac{1}{s}\left(-1 + 2\sum_{k=0}^{\infty}\exp\left(-\left(\frac{2k\pi}{s}\right)^2 at\right)\right)$$

majorizes $\Omega^+(s,\ \xi,\ t)$ and, according to [10, p. 58], has the asymptotic representation

$$\Omega^+(s,\ s,\ t) \sim \frac{1}{2\sqrt{\pi a t}}\left(1 + \exp\left(-\frac{s^2}{4at}\right)\right) \tag{12}$$

for $t \to 0$, $t > 0$. Therefore, to prove that the DS $\tilde{\Omega}^+$ is well-posed relative to the pair $H^\alpha[0,\ b]$ and $L[0,\ b]$ it is sufficient to note that the operator

$$Py = \frac{1}{2\sqrt{a\pi}}\frac{d}{dt}\int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau,$$

acting from the space $H^\alpha[0,\ b]$ into the space $L[0,\ b]$, is bounded. As is known [11], the boundedness of P follows from the expression

$$Py = \frac{1}{2\sqrt{a\pi}}\left(\frac{y(t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{y(t)-y(\tau)}{\sqrt{(t-\tau)^3}}\,d\tau\right).$$

We note that, relative to the pair of Banach spaces $H^\alpha[0,\ b]$ and $H^1[0,\ b]$, natural in the treatment of experimental data, the inverse system $\Psi^-$ no longer has the property of being well-posed.

Thus, if $T_0^-(x) = 0$ and $f^- = 0$, then in calculating q for short times one can, in accordance with (12), use the asymptotic equation

$$q(t) \sim -2\lambda\left(\frac{d}{dt}\int_0^t \frac{y(\tau)\,d\tau}{2\sqrt{a\pi(t-\tau)}} + \int_0^t K(t-\tau)\,y(\tau)\,d\tau\right),$$

where

$$K(t) = \frac{d}{dt}\left(\frac{1}{2\sqrt{a\pi t}}\exp\left(-\frac{s^2}{4at}\right)\right) = \frac{1}{8a\sqrt{a\pi t^3}}\left(\frac{s^2}{t} - 2a\right)\exp\left(-\frac{s^2}{4at}\right).$$

To determine the quantity q(t) from Eqs. (10) and (11) in the general case it is sufficient to have the odd parts $T_0^-(x)$ and $f^-(x,\ t)$, relative to the axis $x = s/2$, of the initial temperature distribution of the plate and the power of the internal sources as the additional information.

In conclusion, we give the representation, following from the block diagram of the DS $\Omega^{-1}$, of the Green's function $\Omega^{-1}(x,\ \xi,\ t)$ of the nonclassical boundary-value problem (1), (2):

$$\Omega^{-1}(x,\ \xi,\ t) = \Omega^+(x,\ \xi,\ t) + \Psi^-(x,\ \xi,\ t) + \int_0^t \Omega^+(x,\ 0,\ \tau)\,\Psi_x^-(0,\ \xi,\ t-\tau)\,d\tau.$$

## NOTATION

T, temperature field; x, spatial coordinate; t, time; $a$, coefficient of thermal diffusivity; $\lambda$, coefficient of thermal conductivity; $\lambda/a$ f, power density of volumetric heat sources; $\Omega$, dynamic system (DS) describing the direct problem of heat conduction; q, heat-flux density at the wall x = s (input of DS $\Omega$); y(t) = T(s, t) − T(0, t), output of DS $\Omega$; H, space of states (temperature fields) of DS $\Omega$; $H^{\alpha}[0, b]$, Banach space of functions defined on the segment [0, b] and satisfying the Hölder condition; L[0, b], space of summable functions.

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